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KALMAN FILTERING AND RICCATI EQUATIONS FOR DESCRIPTOR SYSTEMS

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Kalman filtering and Riccati equations for descriptor systems

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Absract The theory of Kalman filtering is extended to the case of systems with descriptor dynamics. Explicit expressions are obtained for this *descriptor Kalman filter* allowing for the possible singularity of the observation noise covariance. Asymptotic behavior of the filter in the time-invariant case is studied; in particular, a method for constructing the solution of the algebraic *descriptor Riccati equation* is presented.

Le filtre de Kalman et les équations de Riccati pour les systèmes implicites

Résumé Dans ce rapport, nous généralisons la théorie du filtrage de Kalman au cas des systèmes à dynamique implicite. Nous obtenons des formules explicites pour ce *filtre de Kalman implicite* sans aucune hypothèse de non-singularité sur la covariance du bruit de l'observation. De plus, nous étudions la réponse asymptotique du filtre dans le cas des systèmes autonomes, en particulier, on présente une méthode pour la construction de la solution de l'équation algébrique de Riccati.

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1 Introduction

In this paper, we consider the estimation problem for the following system

$$E_{k+1}x(k+1) = A_k x(k) + u(k), \quad k = 0, 1, 2, \ldots \tag{1.1}$$

$$y(k+1) = C_k x(k+1) + r(k), \quad k = 0, 1, 2, \ldots \tag{1.2}$$

where matrices E_{k+1} and A_k are $l_k \times n_k$, and C_k is $p_k \times n_k$; u and r are zero-mean, Gaussian vector sequences with

$$\mathcal{M}\left[\begin{pmatrix} u(k) \\ r(k) \end{pmatrix}\begin{pmatrix} u(j) \\ r(j) \end{pmatrix}^T\right] = \begin{pmatrix} Q_k & S_k \\ S_k^T & R_k \end{pmatrix}\delta(k,j) \tag{1.3}$$

where $\mathcal{M}(.)$ denotes the mean and $\delta(k,j) = 1$ if $k = j$ and 0 otherwise. We suppose that an a priori estimate of $x(0)$ is available.

Since no particular assumption is made about the E_k and A_k matrices, (1.1) does not, in general, specify completely the sequence x. This means that x cannot be thought of as a stochastic process. The point of view that we have adopted in this paper, is to consider x as a sequence of unknown vectors and consider both (1.1) and (1.2) as observations of this unknown sequence. We show that this point of view is consistent with the usual formulation of the Kalman filter in the case of Gauss-Markov processes.

We start, in Section 2, by introducing the maximum likelihood estimation and its connection with Bayesian estimation. In section 3, we derive the descriptor Kalman filter. The results are specialized to the time-invariant case and asymptotic properties of the filter are studied in Section 4. The construction of the steady state descriptor Kalman filter is examined in Section 5. As in the classical case, there exists a dual control problem for the descriptor Kalman filter; this control problem is presented in Section 6.

2 Maximum Likelihood Estimation

2.1 Maximum likelihood versus Bayesian estimation

Let x be unknown constant parameter vector and let z be an observation of x. The if $P(z|x)$ denotes the probability density function of z parameterized by x, the maximum likelihood (ML) estimate $\hat{x}$ based on observation z satisfies

$$p(z|\hat{x}_{ML}) \geq p(z|x) \quad \text{for all } x. \tag{2.1}$$

In the linear Gaussian case, i.e. when

$$z = Lx + v \tag{2.2}$$

where v is a zero-mean, Gaussian random vector with variance R, and L a full column-rank matrix, $\hat{x}_{ML}$ can be obtained by noting that

$$\frac{\partial}{\partial x}\ln(p(z|x))|_{x=\hat{x}_{ML}} = 0. \tag{2.3}$$

Since v is Gaussian, so is z and

$$p(z|x) = \alpha\exp(-(z - Lx)^T R^{-1}(z - Lx)/2) \tag{2.4}$$

where α is a normalization constant. From (2.3) and (2.4), we can see that

$$\hat{x}_{ML} = (L^T R^{-1} L)^{-1} L^T R^{-1} z. \tag{2.5}$$

The error variance associated with this estimate is given by

$$P_{ML} = \mathcal{M}[(x - \hat{x}_{ML})(x - \hat{x}_{ML})^T] = (L^T R^{-1} L)^{-1}. \tag{2.6}$$

To see how the ML estimation method ties in with te Bayesian estimation method, consider the observation (2.2) and suppose that x is not an unknown vector but a Gaussian random vector with known mean $\bar{x}$ and variance P_x. Then the Bayesian estimate $\hat{x}_B$ of x based on observation z is

$$\hat{x}_B = P_B(L^T R^{-1} z + P_x^{-1} \bar{x}) \tag{2.7}$$

where P_B is the covariance of the estimation error:

$$P_B = \mathcal{M}[(x - \hat{x}_B)(x - \hat{x}_B)^T] = (L^T R^{-1} L + P_x^{-1})^{-1}. \tag{2.8}$$

Note that if we let

$$P_x^{-1} = 0, \tag{2.9}$$

the maximum likelihood and the Bayesian estimates and estimation errors are identical.

The maximum likelihood estimation technique can also be used when an a priori estimate of x exists. Specifically, any linear Gaussian Bayesian estimation problem can be formulated as a maximum likelihood estimation problem. Consider the Bayesian problem stated above. This problem can be converted into a maximum likelihood estimation problem if we consider the a priori statistics of x as an extra observation, i.e. consider the following ML estimation problem

$$\begin{pmatrix} z \\ \bar{x} \end{pmatrix} = \begin{pmatrix} L \\ I \end{pmatrix} x + \begin{pmatrix} v \\ w \end{pmatrix} \tag{2.10}$$

where w is a zero-mean, Gaussian random vector, independent of v and with variance P_x. Applying expressions (2.5) and (2.6) to this problem, we obtain the following

$$\hat{x}_{ML} = P_{ML}(L^T R^{-1} z + P_x^{-1} \bar{x}) \tag{2.11}$$

where

$$P_{ML} = (L^T R^{-1} L + P_x^{-1})^{-1} \tag{2.12}$$

which are exactly the Bayesian estimate and estimation error covariance (2.7), (2.8). Thus it is possible to transform any linear Gaussian Bayesian estimation problem into an ML problem by transforming the a priori statistics of x into an observation.

2.2 The case of perfect observation

In the previous section, we considered the case where R, i.e. the variance of the observation noise, is positive definite. If R is not invertible, it is clear that (2.5) and (2.6) cannot be used. In this case, there is a projection of z which is known perfectly, and to obtain the ML estimate, we have to identify this projection.

Consider the ML estimation problem (2.2). Let T be a matrix such that

$$TRT^T = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{R} \end{pmatrix} \tag{2.13}$$

where $\tilde{R}$ is a positive definite matrix. Then (2.2) can be expressed as follows

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} L_1 \\ L_2 \end{pmatrix} x + \begin{pmatrix} 0 \\ \tilde{v} \end{pmatrix} \tag{2.14}$$

where

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = Tz, \quad \begin{pmatrix} L_1 \\ L_2 \end{pmatrix} = TL, \tag{2.15}$$

and $\tilde{v}$ is a zero-mean, Gaussian random vector with invertible variance $\tilde{R}$. Now let S be an invertible matrix such that

$$LS^{-1} = \begin{pmatrix} L_{11} & 0 \end{pmatrix} \tag{2.16}$$

where L_{11} has full column rank. Then if we let

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = Sx, \tag{2.17}$$

(2.14) can be expressed as

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \tilde{v} \end{pmatrix}. \tag{2.18}$$

Finally note that since L_{11} has full rank, it has a left inverse. Let L_{11}^{-L} denote a left inverse of L_{11}, then by premultiplying (2.18) by

$$W = \begin{pmatrix} I & 0 \\ -L_{21}L_{11}^{-L} & I \end{pmatrix} \tag{2.19}$$

we obtain the following

$$\begin{pmatrix} z_1 \\ \tilde{z}_2 \end{pmatrix} = \begin{pmatrix} L_{11} & 0 \\ 0 & L_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \tilde{v} \end{pmatrix} \tag{2.20}$$

where

$$\tilde{z}_2 = z_2 - L_{21}L_{11}^{-L}z_1. \tag{2.21}$$

The vector x_1 is the portion of x which is perfectly observed. Clearly,

$$(\hat{x}_1)_{ML} = L_{11}^{-L}z_1, \tag{2.22}$$

and x_2 can be estimated from the results of the previous section,

$$(\hat{x}_2)_{ML} = (P_2)_{ML}L_{22}^T\tilde{R}^{-1}\tilde{z}_2, \tag{2.23}$$

where

$$(P_2)_{ML} = (L_{22}^T\tilde{R}^{-1}L_{22})^{-1}. \tag{2.24}$$

By combining (2.22) and (2.23), we see that

$$\begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix}_{ML} = \begin{pmatrix} L_{11}^{-L} & 0 \\ 0 & (L_{22}^T\tilde{R}^{-1}L_{22})^{-1}L_{22}^T\tilde{R}^{-1} \end{pmatrix} \begin{pmatrix} z_1 \\ \tilde{z}_2 \end{pmatrix}. \tag{2.25}$$

The ML estimation error covariance P_{ML} and the ML estimate $\hat{x}_{ML}$ are then given by

$$P_{ML} = \begin{pmatrix} 0 & 0 \\ 0 & (L_{22}^T\tilde{R}^{-1}L_{22})^{-1} \end{pmatrix}, \tag{2.26}$$

$$\hat{x}_{ML} = S^{-1}\begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix}_{ML}. \tag{2.27}$$

The above procedure allows us to compute the ML estimate when R is not invertible, however, it does not allow us to express this estimate in a simple closed form expression. The following result which is proved in Appendix A, allows us to express this estimate in terms of a limit which, even though it is not useful for computing the ML estimate, it is useful for analysis.

Lemma 2.1 *Consider the ML estimation problem (2.2) with R possibly singular, then*

$$P_{ML} = \lim_{\epsilon \to 0^+} (L^T(R+\epsilon Q)^{-1}L)^{-1} \tag{2.28}$$

$$\hat{x}_{ML} = \lim_{\epsilon \to 0^+} ((L^T(R+\epsilon Q)^{-1}L)^{-1}L^T(R+\epsilon Q)^{-1})z \tag{2.29}$$

where Q is any positive semi-definite matrix for which the $R+\epsilon Q$ is positive definite for $\epsilon \geq 0$. In particular, we can take $Q = I$.

Note that P_{ML} does not depend on the choice of Q, however,

$$\lim_{\epsilon \to 0^+} (L^T(R+\epsilon Q)^{-1}L)^{-1}L^T(R+\epsilon Q)^{-1}$$

may reflecting the non-unicity in the choice of L_{11}^{-L}.

It is easy to verify that P_{ML} and $\hat{x}_{ML}$ can also be expressed as follows

$$P_{ML} = \lim_{\epsilon \to 0^+} -\begin{pmatrix} 0 & I \end{pmatrix} \begin{pmatrix} R+\epsilon Q & L \\ L^T & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ I \end{pmatrix} \tag{2.30}$$

$$\hat{x}_{ML} = \lim_{\epsilon \to 0^+} \begin{pmatrix} 0 & I \end{pmatrix} \begin{pmatrix} R+\epsilon Q & L \\ L^T & 0 \end{pmatrix}^{-1} \begin{pmatrix} I \\ 0 \end{pmatrix} z \tag{2.31}$$

Which in the case there is no redundant perfect information, i.e., $(L \quad R)$ has full row rank or equivalently, L_{11} in (2.18) is square, simplify as follows (see Lemma 2.2)

$$P_{ML} = -\begin{pmatrix} 0 & I \end{pmatrix} \begin{pmatrix} R & L \\ L^T & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ I \end{pmatrix} \tag{2.32}$$

$$\hat{x}_{ML} = \begin{pmatrix} 0 & I \end{pmatrix} \begin{pmatrix} R & L \\ L^T & 0 \end{pmatrix}^{-1} \begin{pmatrix} I \\ 0 \end{pmatrix} z. \tag{2.33}$$

Expressions (2.32) and (2.33) are used in the next section for deriving a closed-form expression for the descriptor Kalman filter.

Lemma 2.2 *let R be positive semi-definite and L a full column rank matrix. Then if $[R \quad L]$ has full row rank, then*

$$\begin{pmatrix} R & L \\ L^T & 0 \end{pmatrix}$$

is invertible.

Proof Suppose

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} R & L \\ L^T & 0 \end{pmatrix} = 0 \tag{2.34}$$

then

$$xR + yL^T = 0 \tag{2.35}$$

and

$$xL = 0. \tag{2.36}$$

If we now take the conjugate transpose of (2.36) and multiply it on the left by y we get

$$yL^T x^T = 0 \tag{2.37}$$

which after substitution in (2.35) postmultiplied by x^T yields

$$xRx^T = 0 \tag{2.38}$$

which since R is positive semi-definite gives $xR = 0$. But we also have $xL = 0$ and so $x = 0$. It also implies that $yL^T = 0$ which since L is full rank implies that $y = 0$. □

3 Descriptor Kalman filter

Consider the standard Kalman filtering problem for causal Gauss-Markov process:

$$\begin{aligned} x(k+1) &= A_k x(k) + u(k), \quad k = 0,1,2,\ldots && (3.1)\\ y(k+1) &= C_k x(k+1) + r(k), \quad k = 0,1,2,\ldots && (3.2) \end{aligned}$$

where u and r are white Gaussian sequences with

$$\mathcal{M}\left[\begin{pmatrix} u(k) \\ r(k) \end{pmatrix}\begin{pmatrix} u(j) \\ r(j) \end{pmatrix}^T\right] = \begin{pmatrix} Q_k & S_k \\ S_k^T & R_k \end{pmatrix}\delta(k,j). \tag{3.3}$$

The initial state $x(0)$ is also Gaussian with mean $\bar{x}_0$, variance P_0 and independent of u and r. The Kalman filter for this problem consists of sequentially computing the Bayesian estimate $\hat{x}_B(k)$ of the state $x(k)$ based on observations (3.2) up to time $k-1$. The usual derivation of the Kalman filter equations is the construction of the Bayesian estimate of the random process x based on past information y. But the Kalman filter can also be obtained using the ML formulation. In particular, we consider x to be an unknown sequence and convert all the dynamics equations (3.1) and the a priori statistics of $x(0)$ into observations. The ML estimation problem is then

$$\begin{aligned} 0 &= x(k+1) - A_k x(k) - u(k), \quad k = 0,1,2,\ldots && (3.4)\\ y(k+1) &= C_k x(k+1) + r(k), \quad k = 0,1,2,\ldots && (3.5)\\ \bar{x}_0 &= x(0) + \nu && (3.6) \end{aligned}$$

where ν is a Gaussian random vector, independent of u and r, with variance P_0. Here, all of the left hand sides of (3.4)-(3.6) should be considered as measurements, with $-u(k)$, $r(k)$ and ν playing the roles of measurement noises. A question that arises at this point is whether $\hat{x}_B(k)$ is equal to the ML estimate of $x(k)$ based on (3.6), (3.5) for $1 \leq k \leq j-1$ and (3.4) for all k, or, (3.6), and (3.5) and (3.4) for $0 \leq k \leq j-1$. The answer is that *both* of these ML estimates yield the same result. It is straightforward to check, using the results of the previous section, that future dynamics, given observations up to the present time, do not supply any information regarding the present state. To see this, consider the "one step in the future" dynamics equation for $x(j)$:

$$0 = x(j+1) - A_j x(j) - u(j). \tag{3.7}$$

Given observations (3.5) only up to time $k = j-1$, $x(j+1)$ is completely unknown which clearly implies that (3.7) cannot supply any information regarding the value of $x(j)$. Since (3.7), with j replaced with $j+1$, does not contain any information regarding $x(j+1)$ either, then by induction we can see that no future dynamic contains information regarding $x(j)$. This, in fact, is closely related to the Markovian nature of the process x in the original formulation of the problem.

This ML formulation of the optimal filtering problem (Kalman filter) can be extended to the descriptor system (1.1)-(1.2). We consider x as an unknown sequence and convert all dynamics equations (1.1) and the a priori information on $x(0)$ into observations. The ML estimation problem is then

$$\begin{aligned} 0 &= E_{k+1}x(k+1) - A_k x(k) - u(k), \quad k = 0,1,2,... && (3.8)\\ y(k+1) &= C_k x(k+1) + r(k), \quad k = 0,1,2,... && (3.9)\\ \bar{x}_0 &= x(0) + \nu && (3.10) \end{aligned}$$

where ν is a Gaussian random vector, independent of u and r, with variance P_0.

The difference here with the previous case is that (3.8) when $k = j$ does indeed contain information about $x(j)$ when E_{j+1} is not invertible even if $x(j+1)$ is completely unknown. Specifically, (3.8) contains information about the projection of $A_j x(j)$ which lies in the null-space of E_{j+1}. In general, the situation is even more complex because $x(j+1)$ is not completely unknown because of future dynamics. Thus, the optimal estimate of $x(j)$ based on observations (3.9), up to $j-1$, and observations (3.8) up to $j-1$, in general differs from the optimal estimate of $x(j)$ based on observations (3.9), up to $j-1$, and observations (3.8), for all k. For example consider the case where

$$E = B = 0, \quad A = C = I \tag{3.11}$$

with an a priori estimate $\bar{x}_0$ of $x(0)$ with an associated positive-definite error variance P_0. In this case, clearly the only possibility is that $x(j) = 0$ for all j, however, based on the observations (3.8) up to $j-1$ and observations (3.9), up to $j-1$, one can check that

$$\hat{x}_{ML}(j) = y(j), \quad j = 1,2,... \tag{3.12}$$

So in formulating the Kalman filter for descriptor systems, we have to decide what we mean by the filtered estimate $\hat{x}(j)$ of $x(j)$. If we choose to take into account all future dynamics, we do not get a causal filter because the estimate of $x(j)$ could depend on A_k and E_k for $k > j$. There is a priori no upper bound on how far in the future we have to look.

Other reason for not considering the future dynamics is that when we consider the smoothing problem, we like to have a forward and a backward Kalman filter from which the smoothed estimate can be constructed. In that case, clearly considering all the dynamics for both filters would amount to counting the dynamics equations twice. We shall consider the smoothing problem in a subsequent paper.

Definition 3.1 *The filtered estimate $\hat{x}(j)$ of $x(j)$ in (1.1)-(1.2) is defined as the ML estimate of $x(j)$ based on observations (3.9) and dynamics (observations (3.8)) up to $j-1$ and the a priori information, i.e. (3.10). P_j denotes the error variance associated with this estimate. The estimate $\hat{x}(0) = \bar{x}_0$ with an error variance P_0.*

Note that when $\begin{pmatrix} E_j \\ C_j \end{pmatrix}$ does not have full column rank for some j, some projection of $x(j)$ is completely unobserved and no finite error variance estimate of it can be constructed. Thus we shall assume from here on that $\begin{pmatrix} E_j \\ C_j \end{pmatrix}$ has full column rank for all $j \geq 1$.

Theorem 3.1 *Let $\hat{x}(j)$ denote the filtered estimate of $x(j)$ and P_j the associated error variance for descriptor system (1.1)-(1.2). Then, the filtered estimate $\hat{x}(j+1)$ of $x(j+1)$ and the associated error variance P_{j+1}*

are respectively equal to the ML estimate of $x(j+1)$ and its associated estimation error variance based on the following observations

$$\begin{aligned} y(j+1) &= C_{j+1}x(j+1) + r(j) && (3.13)\\ A_j\hat{x}(j) &= E_{j+1}x(j+1) + w(j) && (3.14) \end{aligned}$$

where $w(j)$ is a Gaussian random vector independent of $r(j+1)$ with variance $A_jP_jA_j^T + Q_j$.

The proof is given in Appendix B.

Theorem 3.1 implies that we can construct the estimate at time $j+1$ from the estimate at time j and $y(j+1)$. This gives us a recursive method for computing $\hat{x}(j)$ and is just the descriptor Kalman filter.

Corollary 3.1 *If past and present observations and dynamics do not supply redundant perfect informations, i.e. when*

$$\begin{pmatrix} A_jP_jA_j^T + Q_j & S_j & E_{j+1} \\ S_j^T & R_j & C_{j+1} \end{pmatrix}$$

has full row rank, then the filtered estimate $\hat{x}(j+1)$ and the corresponding error variance P_{j+1} can be obtained as follows:

$$\hat{x}(j+1) = \begin{pmatrix} 0 & 0 & I \end{pmatrix} \begin{pmatrix} A_jP_jA_j^T + Q_j & S_j & E_{j+1} \\ S_j^T & R_j & C_{j+1} \\ E_{j+1}^T & C_{j+1}^T & 0 \end{pmatrix}^{-1} \begin{pmatrix} A_j\hat{x}(j) \\ y(j+1) \\ 0 \end{pmatrix}. \quad (3.15)$$

$$P_{j+1} = -\begin{pmatrix} 0 & 0 & I \end{pmatrix} \begin{pmatrix} A_jP_jA_j^T + Q_j & S_j & E_{j+1} \\ S_j^T & R_j & C_{j+1} \\ E_{j+1}^T & C_{j+1}^T & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \\ I \end{pmatrix}. \quad (3.16)$$

Note that the condition of Theorem 3.1 may not be fulfilled even in the standard causal case, i.e. when $E_k = I$ if R_k is singular.

4 Time-invariant case

In this section, we study the asymptotic properties of the descriptor Kalman filter in the time-invariant case:

$$\begin{aligned} Ex(k+1) &= Ax(k) + u(k), \quad k = 0,1,2,\ldots && (4.17)\\ y(k+1) &= Cx(k+1) + r(k), \quad k = 0,1,2,\ldots && (4.18) \end{aligned}$$

where matrices E and A are $l \times n$, and C is $p \times n$; u and r are zero-mean, white, Gaussian sequences with variance

$$\mathcal{M}\left[\begin{pmatrix} u(k) \\ r(k) \end{pmatrix} \begin{pmatrix} u(j) \\ r(j) \end{pmatrix}^T\right] = \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \delta(k,j). \quad (4.19)$$

4.1 Stability and convergence of the descriptor Kalman filter

Here, we extend the existing results concerning the stability and the convergence of the standard Kalman filter to the descriptor case.

Definition 4.1 *System (4.17)-(4.18) is called detectable if*

$$\begin{pmatrix} sE - tA \\ C \end{pmatrix}$$

has full column rank for all $(s,t) \neq (0,0)$ *such that* $|s| \geq |t|$.

It is called stabilizable if

$$\begin{pmatrix} sE - tA & Q & S \\ C & S^T & R \end{pmatrix}$$

has full row rank for all $(s,t) \neq (0,0)$ *such that* $|s| \geq |t|$.

Note that our definitions of stabilizability and detectability are consistent with classical definitions when $E = I$ and $R > 0$.

If the system is detectable, we can always find a stable estimator filter. The optimal estimator, i.e. the descriptor Kalman filter, however does not necessarily converge to a stable filter.

Theorem 4.1 *Let (4.17)-(4.18) be detectable, then there exists a stable filter*

$$x_s(k+1) = A_s x_s(k) + K_s y(k+1) \tag{4.20}$$

such that

$$\lim_{k\to\infty} \mathcal{M}[(x(k) - x_s(k))(x(k) - x_s(k))^T] < \infty. \tag{4.21}$$

Proof of Theorem 4.1: We start the proof by showing the following lemma:

Lemma 4.1 *Let (4.17)-(4.18) be detectable, then there exists a left inverse* $(L_e \;\; L_c)$ *of* $\begin{pmatrix} E \\ C \end{pmatrix}$, *i.e.*

$$L_e E + L_c C = I, \tag{4.22}$$

such that $L_e A$ *is stable.*

Proof of Lemma 4.1: Since $\begin{pmatrix} E \\ C \end{pmatrix}$ has full rank, there exist invertible matrices U and V such that

$$UEV = \begin{pmatrix} I & 0 \\ 0 & E_{22} \end{pmatrix} \tag{4.23}$$

$$CV = \begin{pmatrix} 0 & C_2 \end{pmatrix} \tag{4.24}$$

where C_2 has full column rank. If we denote

$$UAV = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \tag{4.25}$$

then detectability of (C, E, A) implies that

$$\begin{pmatrix} sI - A_{11} \\ A_{12} \end{pmatrix}$$

has full column rank for $|s/t| \geq 1$ which means that (A_{11}, A_{21}) is detectable in the classical sense. Thus, there exists a matrix D such that $A_{11} + DA_{12}$ is stable. It is now straightforward to verify that if F is any matrix satisfying

$$FC_2 = \begin{pmatrix} -DE_{22} \\ I \end{pmatrix} \tag{4.26}$$

then

$$\begin{pmatrix} I & D \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & E_{22} \end{pmatrix} + F \begin{pmatrix} 0 & C_2 \end{pmatrix} = I \tag{4.27}$$

and

$$\begin{pmatrix} I & D \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} A_{11} + DA_{21} & A_{12} + DA_{22} \\ 0 & 0 \end{pmatrix} \tag{4.28}$$

which is stable because $A_{11} + DA_{12}$ is stable. Thus by taking

$$L_e = V \begin{pmatrix} I & D \\ 0 & 0 \end{pmatrix} U \tag{4.29}$$

$$L_c = VFU \tag{4.30}$$

the lemma is proved. □

Continuing the proof of the theorem, note that using the above lemma, we can express $x(k+1)$ as follows

$$x(k+1) = L_e Ax(k) + L_c y(k+1) + L_e u(k) - L_c r(k+1) \tag{4.31}$$

where $L_e A$ is stable. If we now define

$$x_s(k+1) = L_e A x_s(k) + L_c y(k+1) \tag{4.32}$$

we can easily see that

$$\lim_{k \to \infty} \mathcal{M}[(x(k) - x_s(k))(x(k) - x_s(k))^T] = P_s \tag{4.33}$$

where P_s is the unique positive semi-definite solution of the Lyapunov equation

$$P_s - (L_e A) P_s (L_e A)^T = \begin{pmatrix} L_e & L_c \end{pmatrix} \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \begin{pmatrix} L_e^T \\ L_c^T \end{pmatrix}. \tag{4.34}$$

The theorem is thus proved. □

Theorem 4.2 *Let (4.17)-(4.18) be detectable. Then the algebraic descriptor Riccati equation*

$$P = \lim_{\epsilon \to o^+} \left[\begin{pmatrix} E^T & C^T \end{pmatrix} \left(\begin{pmatrix} APA^T + Q & S \\ S^T & R \end{pmatrix} + \epsilon I \right)^{-1} \begin{pmatrix} E \\ C \end{pmatrix} \right]^{-1} \tag{4.35}$$

has a positive semi-definite solution.

Proof of Theorem 4.2: We prove the existence of a positive semi-definite solution P to (4.35) by showing that the descriptor Riccati recursion

$$P_{j+1} = \lim_{\epsilon \to o+} \left[\begin{pmatrix} E^T & C^T \end{pmatrix} \left(\begin{pmatrix} AP_jA^T + Q & S \\ S^T & R \end{pmatrix} + \epsilon I \right)^{-1} \begin{pmatrix} E \\ C \end{pmatrix} \right]^{-1} \tag{4.36}$$

with $P_0 = 0$ is monotone increasing and bounded.

To see the boundedness of P_k, consider the stable filter (4.32) with $x_s(0) = \bar{x}_0$. It is then clear that the associated error variance matrices $P_s(k)$ converge asymptotically to P_s the unique solution of (4.34) and that thanks to the optimality of the descriptor Kalman filter, $P_k \leq P_s(k)$.

We show that P_k is monotone increasing by induction. Clearly

$$P_1 \geq P_0 = 0. \tag{4.37}$$

Now suppose that

$$P_j \geq P_{j-1}, \tag{4.38}$$

then

$$\begin{aligned} AP_jA^T + Q &\geq AP_{j-1}A^T + Q, && (4.39)\\ \left(\begin{pmatrix} AP_jA^T + Q & S \\ S^T & R \end{pmatrix} + \epsilon I \right)^{-1} &\leq \left(\begin{pmatrix} AP_{j-1}A^T + Q & S \\ S^T & R \end{pmatrix} + \epsilon I \right)^{-1}, && (4.40)\\ P_{j+1} &\geq P_j. && (4.41) \end{aligned}$$

□

Theorem 4.3 *Let (4.17)-(4.18) be detectable and stabilizable. Then for all initial condition P_0, as k goes to infinity, P_k converges exponentially fast to the unique positive semi-definite solution of the algebraic descriptor Riccati equation. Moreover, the descriptor Kalman filter converges to a stable filter.*

Proof of Theorem 4.3: We have already shown that P_k converges when $P_0 = 0$, so we start by extending this result to the case of arbitrary P_0. Then we prove that the convergence is exponential by showing that the asymptotic filter is stable.

It is straightforward to verify that (4.36) can be expressed as

$$P_{j+1} = (L_jA)P_j(L_jA)^T + \begin{pmatrix} L_j & K_j \end{pmatrix} \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \begin{pmatrix} L_j^T \\ K_j^T \end{pmatrix} \tag{4.42}$$

where

$$\begin{pmatrix} L_j & K_j \end{pmatrix} = \lim_{\epsilon \to 0+}$$

$$\left[\begin{pmatrix} E^T & C^T \end{pmatrix} \left(\begin{pmatrix} AP_jA^T + Q & S \\ S^T & R \end{pmatrix} + \epsilon I \right)^{-1} \begin{pmatrix} E \\ C \end{pmatrix} \right]^{-1} \begin{pmatrix} E^T & C^T \end{pmatrix} \left(\begin{pmatrix} AP_jA^T + Q & S \\ S^T & R \end{pmatrix} + \epsilon I \right)^{-1}. \tag{4.43}$$

The algebraic Riccati equation can now be expressed as

$$P = (LA)P(LA)^T + \begin{pmatrix} L & K \end{pmatrix} \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \begin{pmatrix} L^T \\ K^T \end{pmatrix} \tag{4.44}$$

where

$$\begin{pmatrix} L & K \end{pmatrix} = \lim_{\epsilon \to 0+}$$

$$\left[\begin{pmatrix} E^T & C^T \end{pmatrix}\left(\begin{pmatrix} APA^T+Q & S \\ S^T & R \end{pmatrix}+\epsilon I\right)^{-1}\begin{pmatrix} E \\ C \end{pmatrix}\right]^{-1}\begin{pmatrix} E^T & C^T \end{pmatrix}\left(\begin{pmatrix} APA^T+Q & S \\ S^T & R \end{pmatrix}+\epsilon I\right)^{-1}. \tag{4.45}$$

It is not difficult also to see that

$$LE = I - KC. \tag{4.46}$$

We show that LA is stable using the fact that (4.44) has a positive semi-definite solution. Suppose LA is not stable then there exist a scalar $\lambda \geq 1$ and a complex row vector v such that

$$vLA = \lambda v. \tag{4.47}$$

From (4.44) we get that

$$(1-|\lambda|^2)vPv^H = v\begin{pmatrix} L & K \end{pmatrix}\begin{pmatrix} Q & S \\ S^T & R \end{pmatrix}\begin{pmatrix} L^T \\ K^T \end{pmatrix}v^H \tag{4.48}$$

where $(.)^H$ denotes the conjugate-transpose. Since the right hand side of (4.48) is non-negative and its left hand side, non-positive, we must have

$$v\begin{pmatrix} L & K \end{pmatrix}\begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} = 0. \tag{4.49}$$

But then thanks to (4.46) and (4.47) we get that

$$\lambda vLE = vLA - \lambda vKC. \tag{4.50}$$

From (4.49) and (4.50) follows that

$$v\begin{pmatrix} L & K \end{pmatrix}\begin{pmatrix} \lambda E - A & Q & S \\ C & S^T & R \end{pmatrix} = 0 \tag{4.51}$$

which since $v(L \quad K) \neq 0$ ($vL \neq 0$ thanks to (4.47)) contradicts the stabilizability assumption. Thus LA is stable.

Now we must show that for any arbitrary positive semi-definite P_0, the descriptor Riccati equation converges. We shall first prove this result for the case where P_0 is positive-definite. From (4.42) we get that

$$P_j = (L_{j-1}AL_{j-2}A...L_0A)P_0(L_{j-1}AL_{j-2}A...L_0A)^T + \text{non-negative terms}. \tag{4.52}$$

But P_j is bounded thus thanks to the assumption that P_0 is positive-definite, we get that

$$G_j = L_{j-1}AL_{j-2}A...L_0A \tag{4.53}$$

is bounded. Now let P be a positive semi-definite solution of the algebraic descriptor Riccati equation (4.35), then using expression (4.36) and after some algebra we can show that

$$P_{j+1} - P = (LA)(P_j - P)(L_jA)^T. \tag{4.54}$$

Thus,

$$P_{j+1} - P = (LA)^{j+1}(P_0 - P)G_{j+1}^T, \tag{4.55}$$

but G_j is bounded and LA is stable, which mean that P_j converges to P.

To extend this result to the case where P_0 is only positive semi-definite, simply let P_j^1 represent the sequence of matrices satisfying the descriptor Riccati equation with $P_0^1 = 0$ and P_j^2 the sequence of matrices satisfying the same equation with $P_0^2 \geq P_0$ and $P_0^2 > 0$. Then $P_j^1 \leq P_j \leq P_j^2$ (where P_j denotes the sequence generated by the descriptor Riccati equation with initial condition P_0) and since P_j^1 and P_j^2 converge to P, so does P_j.

Note that we have shown that P_j converges to P from any arbitrary initial condition $P_0 \geq 0$, where P is any positive semi-definite solution of the algebraic descriptor Riccati equation. This clearly implies that P is unique.

Now we have to show that P_j converges exponentially fast for any initial condition $P_0 \geq 0$. For this, we shall show the result for the case where $P_0 > 0$ and the case where $P_0 = 0$. These results can then be extended to the general case by an argument similar to the one used to show convergence.

Let P_0 be positive-definite, then G_j is bounded. Thus, since LA is stable, from (4.55) we can deduce that P_j converges to P exponentially fast at a rate determined by the magnitude of the largest eigenvalue of LA.

Now let P_0 be zero. If for some j, P_j becomes positive definite then exponential convergence follows immediately from the result of the previous case. If P_j never becomes positive definite, using the fact that P_j is monotone increasing, it can be deduced that

$$\mathrm{Im}\,(P_j) \subset \mathrm{Im}\,(P_{j+1}), \quad j \geq 0. \tag{4.56}$$

Thus for some $k > 0$,

$$\mathrm{Im}\,(P_j) \subset \mathrm{Im}\,(P_k), \quad j \geq 0. \tag{4.57}$$

This means that there is a projection of x which is estimated perfectly, and a projection, which after the k^{th} step can only be estimated with a positive definite error variance. We can, without loss of generality assume that

$$x(j) = \begin{pmatrix} x^p(j) \\ x^q(j) \end{pmatrix} \tag{4.58}$$

where $x^p(j)$ is the perfectly observed projection and $x^q(j)$ the "non-perfectly observed projection". In this case,

$$P_j = \begin{pmatrix} 0 & 0 \\ 0 & P_j^q \end{pmatrix} \tag{4.59}$$

where $P_j^q > 0$ if $j \geq k$, and

$$P = \begin{pmatrix} 0 & 0 \\ 0 & P^q \end{pmatrix} \tag{4.60}$$

where $P^q > 0$. Also note that the dynamics of the descriptor Kalman filter L_jA, for $j \geq k$, must be block tri-diagonal:

$$L_jA = \begin{pmatrix} M_j^p & 0 \\ M_j^{pq} & M_j^q \end{pmatrix} \tag{4.61}$$

because $x^p(j+1)$ can only be updated in terms of $x^p(j)$ and noise uncorrupted projection of $y(j+1)$ otherwise, it would not have a zero variance estimate. But we know that

$$P_{k+j+1} = (LA)^{k+j+1}(P_k - P)[L_{k+j}AL_{k+j-1}A...L_kA]^T \tag{4.62}$$

which if we denote by M^q the $(2,2)$-block of LA, yields

$$P^q_{k+j+1} = M^q_{k+j+1}(P^q_k - P^q)[M^q_{k+j}M^q_{k+j-1}\dots M^q_k]^T. \tag{4.63}$$

Note that P^q_k is positive definite, so

$$M^q_{k+j}M^q_{k+j-1}\dots M^q_k$$

is bounded, and since M^q is stable, P^q_j converges exponentially fast. □

Note the stabilizability assumption in particular implies that

$$\begin{pmatrix} E & Q & S \\ C & S^T & R \end{pmatrix}$$

has full row rank which implies that[4]

$$\begin{pmatrix} E & Q + AP_jA^T & S \\ C & S^T & R \end{pmatrix}$$

has full row rank for all $P_j \geq 0$. Thus, the descriptor Kalman filter can be expressed explicitly as indicated in Corollary 3.1. The main results of this section are summerized in the next theorem:

Theorem 4.4 *Let (4.17)-(4.18) be detectable and stabilizable. Then*

1- the descriptor Kalman filter can be expressed as

$$\hat{x}(j+1) = \begin{pmatrix} 0 & 0 & I \end{pmatrix} \begin{pmatrix} AP_jA^T + Q & S & E \\ S^T & R & C \\ E^T & C^T & 0 \end{pmatrix}^{-1} \begin{pmatrix} A\hat{x}(j) \\ y(j+1) \\ 0 \end{pmatrix}, \quad \hat{x}(0) = \bar{x}_0 \tag{4.64}$$

$$P_{j+1} = -\begin{pmatrix} 0 & 0 & I \end{pmatrix} \begin{pmatrix} AP_jA^T + Q & S & E \\ S^T & R & C \\ E^T & C^T & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \\ I \end{pmatrix}, \tag{4.65}$$

2- the error variance matrix P_j converges exponentially to P the unique positive semi-definite solution of the algebraic descriptor Riccati equation

$$P = -\begin{pmatrix} 0 & 0 & I \end{pmatrix} \begin{pmatrix} APA^T + Q & S & E \\ S^T & R & C \\ E^T & C^T & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \\ I \end{pmatrix}, \tag{4.66}$$

3- the descriptor Kalman filter converges to the stable steady state descriptor Kalman filter

$$\hat{x}(j+1) = \begin{pmatrix} 0 & 0 & I \end{pmatrix} \begin{pmatrix} APA^T + Q & S & E \\ S^T & R & C \\ E^T & C^T & 0 \end{pmatrix}^{-1} \begin{pmatrix} A\hat{x}(j) \\ y(j+1) \\ 0 \end{pmatrix}. \tag{4.67}$$

Next section is concerned with the construction of the matrix P.

[4] In general if $[X \;\; Y]$, $Y \geq 0$, has full row rank then $[X, Y + Z]$, $Z \geq 0$, has full row rank because $\operatorname{Ker} Y \subset \operatorname{Ker} Y + Z$.

5 Construction of the steady state filter

In this section, we show that the solution of the algebraic descriptor Riccati equation can be constructed using the eigenvectors and generalized eigenvectors of the pencil:

$$\left\{ \begin{pmatrix} E & -Q & -S \\ C & -S^T & -R \\ 0 & A^T & 0 \end{pmatrix}, \begin{pmatrix} A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & E^T & C^T \end{pmatrix} \right\}. \tag{5.1}$$

We shall assume throughout this section that the system is detectable and stabilizable.

Lemma 5.1 *The pencil (5.1) is regular and has no eigenmode on the unit circle.*

Before proving this lemma, let us introduce the following notation:

$$F = \begin{pmatrix} E \\ C \end{pmatrix}, \quad K = \begin{pmatrix} A \\ 0 \end{pmatrix}, \quad G = \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix}. \tag{5.2}$$

The pencil (5.1) can now be expressed as

$$\left\{ \begin{pmatrix} F & -G \\ 0 & K^T \end{pmatrix}, \begin{pmatrix} K & 0 \\ 0 & F^T \end{pmatrix} \right\} \tag{5.3}$$

and the descriptor Riccati equation as

$$P = -\begin{pmatrix} 0 & I \end{pmatrix} \begin{pmatrix} KPK^T + G & F \\ F^T & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ I \end{pmatrix}. \tag{5.4}$$

Proof of Lemma 5.1: All we need to show is that for all z on the unit circle,

$$\begin{pmatrix} F & -G \\ 0 & K^T \end{pmatrix} + z \begin{pmatrix} K & 0 \\ 0 & F^T \end{pmatrix} \tag{5.5}$$

is invertible. Note that thanks to the detectability assumption which can now be stated in terms of the new notation as: "$sF - tK$ has full column rank for $(s,t) \neq (0,0)$ and $|s| \geq |t|$", we can see that $F + zK$ has full column rank for all z on the unit circle. Now suppose that (5.5) is not invertible, which means that there exist u and v not simultaneously null such that

$$\begin{pmatrix} u & v \end{pmatrix} \begin{pmatrix} F + zK & -G \\ 0 & K^T + zF^T \end{pmatrix} = 0. \tag{5.6}$$

If we now let

$$\Gamma = zK + F \tag{5.7}$$

from (5.6) follows that

$$u\Gamma + vG = 0 \tag{5.8}$$

$$v\Gamma^H = 0. \tag{5.9}$$

If we now multiply (5.8) and (5.9) on the right by v^H and u^H respectively and take the transpose-conjugate of (5.9) and subtract from (5.8), we get

$$vGv^H = 0 \tag{5.10}$$

which since G is symmetric positive semi-definite implies that $vG = 0$. Thus since Γ has full column rank, (5.8) implies that $u = 0$. But we also have that $v(\Gamma^H \quad G) = 0$ which thanks to the stabilizability assumption implies that $v = 0$ contradicting the assumption that u and v are not simultaneously null. □

Lemma 5.2 *The pencil (5.1) has exactly n stable eigenmodes.*

Proof of Lemma 5.2: Let

$$\begin{aligned} p(s,t) &= \det\left(s\begin{pmatrix} F & -G \\ 0 & K^T \end{pmatrix} + t\begin{pmatrix} K & 0 \\ 0 & F^T \end{pmatrix}\right) \\ &= \det\left(s\begin{pmatrix} 0 & K^T \\ F & -G \end{pmatrix} + t\begin{pmatrix} 0 & F^T \\ K & 0 \end{pmatrix}\right) = \det\begin{pmatrix} 0 & sK^T + tF^T \\ sF + tK & -sG \end{pmatrix} \end{aligned} \tag{5.11}$$

then

$$p(t,s) = \det\left(t\begin{pmatrix} 0 & K^T \\ F & -G \end{pmatrix} + s\begin{pmatrix} 0 & F^T \\ K & 0 \end{pmatrix}\right) = \det\begin{pmatrix} 0 & tK^T + sF^T \\ tF + sK & -tG \end{pmatrix}. \tag{5.12}$$

Note that

$$\begin{aligned} &\begin{pmatrix} I & 0 \\ 0 & tI \end{pmatrix}\begin{pmatrix} 0 & sK^T + tF^T \\ sF + tK & -sG \end{pmatrix}\begin{pmatrix} I/t & 0 \\ 0 & I \end{pmatrix} = \\ &\begin{pmatrix} I & 0 \\ 0 & sI \end{pmatrix}\begin{pmatrix} 0 & tK^T + sF^T \\ tF + sK & -tG \end{pmatrix}^T\begin{pmatrix} I/s & 0 \\ 0 & I \end{pmatrix} \end{aligned} \tag{5.13}$$

and so

$$t^{l+p}p(s,t)t^{-n} = s^{l+p}p(t,s)s^{-n}, \tag{5.14}$$

so

$$t^{l+p-n}p(s,t) = s^{l+p-n}p(t,s). \tag{5.15}$$

If we denote the number of zero eigenmodes by δ_0, stable but non-zero eigenmodes by δ_s, unstable eigenmodes by δ_u and infinite eigenmodes by δ_∞, from (5.15) and the fact that there are no eigenmodes on the unit circle, we conclude that

$$\begin{aligned} \delta_s &= \delta_u && (5.16) \\ \delta_\infty - \delta_0 &= l + p - n. && (5.17) \end{aligned}$$

Finally noting that

$$\delta_0 + \delta_s + \delta_u + \delta_\infty = n + l + p \tag{5.18}$$

we get that the number of stable eigenmodes $\delta_0 + \delta_s = n$. □

Theorem 5.1 *Let the columns of*

$$\begin{pmatrix} X \\ Y_1 \\ Y_2 \end{pmatrix}$$

form a basis for the eigenspace of the pencil (5.1) associated with its n stable eigenmodes, i.e.

$$\begin{pmatrix} E & -Q & -S \\ C & -S^T & -R \\ 0 & A^T & 0 \end{pmatrix}\begin{pmatrix} X \\ Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & E^T & C^T \end{pmatrix}\begin{pmatrix} X \\ Y_1 \\ Y_2 \end{pmatrix} J \tag{5.19}$$

where J is stable. Then, P, the unique positive semi-definite solution of the algebraic Riccati equation (4.66) is given by

$$P = X(E^T Y_1 + C^T Y_2)^{-1}. \tag{5.20}$$

Proof of Theorem 5.1: Using notation (5.2) and by letting

$$Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \tag{5.21}$$

we must show that

$$P = X(FY)^{-1}. \tag{5.22}$$

Consider perturbing G as follows

$$G_\epsilon = G + \epsilon I \tag{5.23}$$

and denote the matrix of eigenvectors and generalized eigenvectors associated with the stable eigenmodes of the perturbed pencil by

$$\begin{pmatrix} X_\epsilon \\ Y_\epsilon \end{pmatrix}$$

and the matrix of eigenmodes by J_ϵ:

$$\begin{pmatrix} F & -G_\epsilon \\ 0 & K^T \end{pmatrix} \begin{pmatrix} X_\epsilon \\ Y_\epsilon \end{pmatrix} = \begin{pmatrix} K & 0 \\ 0 & F^T \end{pmatrix} \begin{pmatrix} X_\epsilon \\ Y_\epsilon \end{pmatrix} J_\epsilon. \tag{5.24}$$

It is well know that as ϵ goes to zero, X_ϵ and Y_ϵ converge respectively to X and Y (J_ϵ may or may not converge to J depending on the eigenstructure of J).

Now suppose that X_ϵ is invertible for $\epsilon > 0$. Then we like to show that $Y_\epsilon X_\epsilon^{-1}$ is real valued. Note that the Jordan blocks of J_ϵ that correspond to complex eigenmodes, are in complex-conjugate pairs and thus J_ϵ and J_ϵ^* (where $(.)^*$ denotes complex conjugate) are similar, i.e. there exists an invertible matrix W such that

$$J_\epsilon = W^{-1} J_\epsilon^* W. \tag{5.25}$$

Thus by using this fact and taking the complex conjugate of both sides of (5.24) and post-multiplication by W, we obtain

$$\begin{pmatrix} F & -G_\epsilon \\ 0 & K^T \end{pmatrix} \begin{pmatrix} X_\epsilon^* W \\ Y_\epsilon^* W \end{pmatrix} = \begin{pmatrix} K & 0 \\ 0 & F^T \end{pmatrix} \begin{pmatrix} X_\epsilon^* W \\ Y_\epsilon^* W \end{pmatrix} J_\epsilon \tag{5.26}$$

which implies that columns of

$$\begin{pmatrix} X_\epsilon^* W \\ Y_\epsilon^* W \end{pmatrix}$$

form a basis for the space spanned by the stable eigenvectors and generalized eigenvectors of the perturbed pencil and thus, for some invertible matrix V, we must have

$$\begin{pmatrix} X_\epsilon^* W \\ Y_\epsilon^* W \end{pmatrix} V = \begin{pmatrix} X_\epsilon \\ Y_\epsilon \end{pmatrix} \tag{5.27}$$

which implies that

$$Y_\epsilon X_\epsilon^{-1} = Y_\epsilon^* (X_\epsilon^*)^{-1} = (Y_\epsilon X_\epsilon^{-1})^*. \tag{5.28}$$

Thus, if X_ϵ is invertible, $Y_\epsilon X_\epsilon^{-1}$ is real valued.

Let us now show that X_ϵ is invertible. From (5.24) we get

$$\begin{aligned} FX_\epsilon - G_\epsilon Y_\epsilon &= KX_\epsilon J_\epsilon && (5.29) \\ K^T Y_\epsilon &= F^T Y_\epsilon J_\epsilon. && (5.30) \end{aligned}$$

If we premultiply (5.29) by Y_ϵ^H we get

$$Y_\epsilon^H F X_\epsilon = Y_\epsilon^H G_\epsilon Y_\epsilon + Y_\epsilon^H K X_\epsilon J_\epsilon. \tag{5.31}$$

Now if we take the conjugate-transpose of (5.30) and use it in (5.31) we get

$$Y_\epsilon^H F X_\epsilon = Y_\epsilon^H G_\epsilon Y_\epsilon + J_\epsilon^H Y_\epsilon^H F X_\epsilon J_\epsilon. \tag{5.32}$$

But (5.32) is a Lyapunov equation with J_ϵ strictly stable and so $Y_\epsilon^H F X_\epsilon$ is Hermitian positive-semi definite. So if we let v be any matrix such that $X_\epsilon v = 0$, by pre and post-multiplying (5.32) by v^H and v respectively, we get

$$v^H Y_\epsilon^H G_\epsilon Y_\epsilon v = 0 \tag{5.33}$$

which since G_ϵ is positive-definite, implies that $Y_\epsilon v = 0$. But this is a contradiction because

$$\begin{pmatrix} X_\epsilon \\ Y_\epsilon \end{pmatrix}$$

has full column rank. Thus, X_ϵ, for small enough $\epsilon > 0$, is invertible. Note that $X(0) = X$ is not necessarily invertible.

Let us now show that $F^T Y_\epsilon$ is invertible. Suppose that $F^T Y_\epsilon$ is not invertible, in that case, from the Lyapunov equation (5.32) we know that there exists an eigenvector w of J_ϵ such that

$$G_\epsilon Y_\epsilon w = 0. \tag{5.34}$$

If we now multiply (5.29) on the right by w and use the fact that $J_\epsilon w = \lambda w$ for some $\lambda < 1$, we get that

$$F X_\epsilon w = \lambda K X_\epsilon w \tag{5.35}$$

contradicting the detectability assumption.[5] Thus, $F^T Y_\epsilon$ is invertible. Note in particular that since we have not used the invertibility of G_ϵ, $F^T Y(0) = F^T Y$ is also invertible.

Now, let us solve for J_ϵ in (5.30) and substitute it in (5.29) and factor Y_ϵ as follows

$$F X_\epsilon = [G_\epsilon + K X_\epsilon (F^T Y_\epsilon)^{-1} K^T] Y_\epsilon \tag{5.36}$$

from which we get

$$(F^T Y_\epsilon) X_\epsilon^{-1} = F^T (G_\epsilon + K X_\epsilon (F^T Y_\epsilon)^{-1} K^T)^{-1} F \tag{5.37}$$

which implies that

$$X_\epsilon (F^T Y_\epsilon)^{-1} = [F^T (G_\epsilon + K X_\epsilon (F^T Y_\epsilon)^{-1} K^T)^{-1} F]^{-1} \tag{5.38}$$

which in turn implies that

$$P(\epsilon) = X_\epsilon (F^T Y_\epsilon)^{-1} \tag{5.39}$$

satisfies the perturbed algebraic descriptor Riccati equation

$$P(\epsilon) = -\begin{pmatrix} 0 & I \end{pmatrix} \begin{pmatrix} K P(\epsilon) K^T + G_\epsilon & F \\ F^T & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ I \end{pmatrix}. \tag{5.40}$$

Matrix $X_\epsilon (F^T Y_\epsilon)^{-1}$ is real-valued and positive-definite: it is invertible because X_ϵ and $F^T Y_\epsilon$ are invertible; it is real-valued because

$$X_\epsilon (F^T Y_\epsilon)^{-1} = (F^T Y_\epsilon X_\epsilon^{-1})^{-1} \tag{5.41}$$

[5] Detectability and stabilizability are generic properties and thus conserved under small enough perturbations.

and we have already shown that $Y_\epsilon X_\epsilon^{-1}$ is real-valued; it is positive semi-definite because $Y_\epsilon^H F X_\epsilon$ is positive semi-definite and

$$X_\epsilon(F^T Y_\epsilon)^{-1} = [(Y_\epsilon^H F)^{-1}] Y_\epsilon^H F X_\epsilon [(Y_\epsilon^H F)^{-1}]^H. \tag{5.42}$$

Now let us show that the solution $P(\epsilon)$ of the perturbed algebraic descriptor Riccati equation (5.40) converges to P as ϵ goes to zero. For that consider the perturbed descriptor Riccati equation

$$P_{j+1}(\epsilon) = -\begin{pmatrix} 0 & I \end{pmatrix} \begin{pmatrix} K P_j(\epsilon) K^T + G_\epsilon & F \\ F^T & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ I \end{pmatrix} \tag{5.43}$$

with $P_0(\epsilon) = P$. From the results of the previous section, we know that $P_j(\epsilon)$ converges exponentially fast to $P(\epsilon)$ as long as the perturbed system remains detectable and stabilizable. Thus there exists $\epsilon_0 > 0$ such that for all ϵ, $0 \leq \epsilon \leq \epsilon_0$, $P_j(\epsilon)$ converges exponentially fast to $P(\epsilon)$ with a rate λ_ϵ. If we let

$$\lambda = \min_{0 \leq \epsilon \leq \epsilon_0} \lambda_\epsilon \tag{5.44}$$

we can see that for $0 \leq \epsilon \leq \epsilon_0$, $P_j(\epsilon)$ converges to $P(\epsilon)$ at least with an exponential rate λ. Thus, $P_j(\epsilon)$ converges uniformly to $P(\epsilon)$. Also note that $P_j(\epsilon)$ is a continuous function of ϵ. But the uniform limit of continuous functions is continuous and so $P(\epsilon)$ is continuous which implies that

$$\lim_{\epsilon \to 0+} P(\epsilon) = P(0) = P. \tag{5.45}$$

Finally, using (5.45) and the fact that X_ϵ and Y_ϵ converge to X and Y, and that $F^T Y$ is invertible, from (5.39), by taking limit as ϵ goes to zero, we get that $P = X(F^T Y)^{-1}$. □

6 Dual control problem

Consider the following optimal control problem:

$$E_{k+1} x(k+1) = A_k x(k) + B_{k+1} u(k+1), \quad x(0) : \text{given} \tag{6.1}$$

$$J = \sum_{j=1}^{N} \begin{pmatrix} x(j)^T & u(j)^T \end{pmatrix} \begin{pmatrix} Q_j & S_j \\ S_j^T & R_j \end{pmatrix} \begin{pmatrix} x(j) \\ u(j) \end{pmatrix}. \tag{6.2}$$

Since x is not in general completely specified in terms of u, the minimization is done over the "bitrajectory" $\{x, u\}$ (see [3]).

This problem can be expressed as follows

$$F_{k+1} \xi(k+1) = K_k \xi(k), \quad \xi(0) : \text{given} \tag{6.3}$$

with

$$J = \sum_{j=1}^{N} \xi(j)^T G_j \xi(j) \tag{6.4}$$

where

$$\begin{aligned} F_k &= \begin{pmatrix} E_k & -B_k \end{pmatrix} & (6.5) \\ K_k &= \begin{pmatrix} A_k & 0 \end{pmatrix} & (6.6) \\ G_k &= \begin{pmatrix} Q_k & S_k \\ S_k^T & R_k \end{pmatrix}. & (6.7) \end{aligned}$$

Use dynamic programming approach (as in Bernhard *et al.*, but more general). Let

$$V_j(\xi(j),...,\xi(N)) = \sum_{k=j}^{N} \xi(k)^T G_k \xi(k) \tag{6.8}$$

and

$$V_j^*(\xi(j)) = \min_{\xi(j+1),...,\xi(N)} V_j(\xi(j),...,\xi(N)). \tag{6.9}$$

It is clear that

$$V_N^*(\xi(N)) = \xi(N)^T G_N \xi(N) \tag{6.10}$$

and that

$$V_{N-1}^*(\xi(N-1)) = \xi(N-1)^T G_{N-1}\xi(N-1) + \min_{\xi(N)} \xi(N)^T G_N \xi(N). \tag{6.11}$$

To compute the minimum in (6.11), use the Lagrange multiplier technique: let

$$H = \xi(N)^T G_N \xi(N) + \lambda^T (F_N \xi(N) - K_{N-1}\xi(N-1)). \tag{6.12}$$

Now by letting the partial of H with respect to $\xi(N)$ to zero we get

$$G_N \xi(N)^* + F_N^T \lambda = 0 \tag{6.13}$$

and thus

$$\begin{pmatrix} G_N & F_N^T \\ F_N & 0 \end{pmatrix} \begin{pmatrix} \xi(N)^* \\ \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ K_{N-1}\xi(N-1) \end{pmatrix}. \tag{6.14}$$

Now assuming that $F_N = [E_N \quad -B_N]$ has full row rank which is a necessary assumption to avoid infinite costs (inadmissable states) and assuming that $[G_N \quad F_N^T]$ has full row rank (the equivalent of no perfect redundant information for filtering problem) we get

$$\xi(N)^* = \begin{pmatrix} I & 0 \end{pmatrix} \begin{pmatrix} G_N & F_N^T \\ F_N & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ K_{N-1} \end{pmatrix} \xi(N-1). \tag{6.15}$$

Thus

$$V_{N-1}^*(\xi(N-1)) = \xi(N-1)^T G_{N-1}\xi(N-1) +$$

$$\xi(N-1)^T \begin{pmatrix} 0 & K_{N-1}^T \end{pmatrix} \begin{pmatrix} G_N & F_N^T \\ F_N & 0 \end{pmatrix}^{-1} \begin{pmatrix} G_N & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} G_N & F_N^T \\ F_N & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ K_{N-1} \end{pmatrix} \xi(N-1) \tag{6.16}$$

which after some algebra yields

$$V_{N-1}^*(\xi(N_1)) = \xi(N-1)^T \left(G_{N-1} - \begin{pmatrix} 0 & K_{N-1}^T \end{pmatrix} \begin{pmatrix} G_N & F_N^T \\ F_N & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ K_{N-1} \end{pmatrix} \right) \xi(N-1). \tag{6.17}$$

So in general

$$V_k^*(\xi(k)) = \xi(k)^T \Delta_k \xi(k) \tag{6.18}$$

where

$$\Delta_{k-1} = G_{k-1} - \begin{pmatrix} 0 & K_{k-1}^T \end{pmatrix} \begin{pmatrix} \Delta_k & F_k^T \\ F_k & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ K_{k-1} \end{pmatrix}, \quad \Delta_N = G_N \tag{6.19}$$

and

$$\xi(k)^* = \begin{pmatrix} 0 & I \end{pmatrix} \begin{pmatrix} \Delta_k & F_k^T \\ F_k & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ K_{k-1} \end{pmatrix} \xi(k-1). \tag{6.20}$$

If we now let

$$P_{k-1} = -\begin{pmatrix} 0 & I \end{pmatrix} \begin{pmatrix} \Delta_k & F_k^T \\ F_k & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ I \end{pmatrix} \tag{6.21}$$

which yields

$$\Delta_k = G_k + K_k^T P_k K_k, \tag{6.22}$$

we obtain the dual control filter

$$\begin{pmatrix} x(k)^* \\ u(k)^* \end{pmatrix} = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \end{pmatrix} \begin{pmatrix} Q_k + A_k^T P_k A_k & S_k & E_k^T \\ S_k^T & R_k & -B_k^T \\ E_k & -B_k & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \\ A_{k-1} \end{pmatrix} x(k-1) \tag{6.23}$$

where P_k satisfies the descriptor Riccati equation

$$P_{k-1} = -\begin{pmatrix} 0 & 0 & I \end{pmatrix} \begin{pmatrix} Q_k + A_k^T P_k A_k & S_k & E_k^T \\ S_k^T & R_k & -B_k^T \\ E_k & -B_k & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \\ I \end{pmatrix} \tag{6.24}$$

with final condition

$$P_N = 0. \tag{6.25}$$

Clearly, the optimal cost $J^* = P_0$.

Note that the descriptor Riccati equation (6.24) is similar to the descriptor Riccati equation for the filtering problem. Thus all of the results obtained for the time invariant filtering problem extends trivially to this case. In particular, for the time-invariant, infinite horizon problem

$$Ex(k+1) = Ax(k) + Bu(k+1), \quad x(0) : \text{given} \tag{6.26}$$

$$J = \sum_{j=1}^{\infty} \begin{pmatrix} x(j)^T & u(j)^T \end{pmatrix} \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \begin{pmatrix} x(j) \\ u(j) \end{pmatrix} \tag{6.27}$$

we obtain the following result:

Theorem 6.1 *Suppose*

$$\begin{pmatrix} sE - tA & B \end{pmatrix}$$

and

$$\begin{pmatrix} sE - tA & B \\ Q & S \\ S^T & R \end{pmatrix}$$

have respectively full row and column ranks for all $(s,t) \neq (0,0)$ and $|s| \geq |t|$. Then, the solution to the infinite horizon problem is given by

$$\begin{pmatrix} x(k)^* \\ u(k)^* \end{pmatrix} = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \end{pmatrix} \begin{pmatrix} Q + A^T P A & S & E^T \\ S^T & R & -B^T \\ E & -B & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \\ A \end{pmatrix} x(k-1) \tag{6.28}$$

where P is the unique positive semi-definite solution of the algebraic descriptor Riccati equation

$$P = -\begin{pmatrix} 0 & 0 & I \end{pmatrix} \begin{pmatrix} Q + A^T P A & S & E^T \\ S^T & R & -B^T \\ E & -B & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \\ I \end{pmatrix}. \tag{6.29}$$

Moreover,

$$P = X(EY_1 - BY_2)^{-1} \tag{6.30}$$

where the columns of

$$\begin{pmatrix} X \\ Y_1 \\ Y_2 \end{pmatrix}$$

form a basis for the eigenspace of the pencil

$$\left\{ \begin{pmatrix} E^T & -Q & -S \\ -B^T & -S^T & -R \\ 0 & A & 0 \end{pmatrix}, \begin{pmatrix} A^T & 0 & 0 \\ 0 & 0 & 0 \\ 0 & E & -B \end{pmatrix} \right\}$$

associated with its stable eigenmodes.

7 Conclusion

We have generalized, in this paper, the theory of Kalman filtering to the case of descriptor systems. In particular, we have derived explicit expressions for the filter and studied its asymtotic behavior. The square-root implementation of this filter will be presented in a subsequent paper.

A Proof of Lemma 2.1

First note that the lemma holds when R is non-singular. Now suppose that R is singular and assume without loss of generality that (2.2) has the following structure (as seen above this can always be achieved by a coordinate transformation and premultiplication of (2.2) by some invertible matrix),

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} L_{11} & 0 \\ 0 & L_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \tilde{v} \end{pmatrix} \tag{A.1}$$

where L_{11} has full column rank and $\tilde{v}$ has an invertible variance denoted by $\tilde{R}$. Let

$$Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix}. \tag{A.2}$$

Then, since

$$R = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{R} \end{pmatrix} \tag{A.3}$$

and thanks to the assumption that $Q \geq 0$ and $R + \epsilon Q$ is invertible, we can see that

$$Q_{11} > 0. \tag{A.4}$$

Expression (2.28) can now be expressed as follows

$$
\begin{aligned}
P_{ML} &= \lim_{\epsilon \to 0^+} \left(\begin{pmatrix} L_{11} & 0 \\ 0 & L_{22} \end{pmatrix}^T \left(\begin{pmatrix} 0 & 0 \\ 0 & \tilde{R} \end{pmatrix} + \epsilon \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} \right)^{-1} \begin{pmatrix} L_{11} & 0 \\ 0 & L_{22} \end{pmatrix} \right)^{-1} \\
&= \lim_{\epsilon \to 0^+} \left(\begin{pmatrix} L_{11} & 0 \\ 0 & L_{22} \end{pmatrix}^T \begin{pmatrix} \epsilon Q_{11} & \epsilon Q_{12} \\ \epsilon Q_{21} & \tilde{R} + \epsilon Q_{22} \end{pmatrix}^{-1} \begin{pmatrix} L_{11} & 0 \\ 0 & L_{22} \end{pmatrix} \right)^{-1} . \qquad \text{(A.5)}
\end{aligned}
$$

To evaluate the above expression, we need the following identity:

$$
\begin{pmatrix} A & D \\ C & B \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} + E\Delta^{-1}F & -E\Delta^{-1} \\ -\Delta^{-1}F & \Delta^{-1} \end{pmatrix} \qquad \text{(A.6)}
$$

where $\Delta = B - CA^{-1}D$, $E = A^{-1}D$ and $F = CA^{-1}$. The $(1,1)$-block entry of (A.6) can also be expressed as $(A - DB^{-1}C)^{-1}$. Using (A.6) with the alternate expression for its $(1,1)$-block entry, we get that

$$
\begin{aligned}
&\begin{pmatrix} \epsilon Q_{11} & \epsilon Q_{12} \\ \epsilon Q_{21} & \tilde{R} + \epsilon Q_{22} \end{pmatrix}^{-1} = \\
&\quad \begin{pmatrix} (\epsilon Q_{11} - \epsilon^2 Q_{12}(\tilde{R} + \epsilon Q_{22})^{-1} Q_{21})^{-1} & -Q_{11}^{-1} Q_{12}(\tilde{R} + \epsilon Q_{22} - \epsilon Q_{21} Q_{11}^{-1} Q_{12})^{-1} \\ -(\tilde{R} + \epsilon Q_{22} - \epsilon Q_{21} Q_{11}^{-1} Q_{12})^{-1} Q_{21} Q_{11}^{-1} & (\tilde{R} + \epsilon Q_{22} - \epsilon Q_{21} Q_{11}^{-1} Q_{12})^{-1} \end{pmatrix} \qquad \text{(A.7)}
\end{aligned}
$$

We can simplify the above expression by separating terms of order ϵ and higher. The result is

$$
\begin{pmatrix} \epsilon Q_{11} & \epsilon Q_{12} \\ \epsilon Q_{21} & \tilde{R} + \epsilon Q_{22} \end{pmatrix}^{-1} = \begin{pmatrix} Q_{11}^{-1}/\epsilon + Q_{11}^{-1} Q_{12} \tilde{R}^{-1} Q_{21} Q_{11}^{-1} & -Q_{11}^{-1} Q_{12} \tilde{R}^{-1} \\ -\tilde{R}^{-1} Q_{21} Q_{11}^{-1} & \tilde{R}^{-1} \end{pmatrix} + o(\epsilon). \qquad \text{(A.8)}
$$

Thus we get

$$
\begin{aligned}
P_{ML} &= \lim_{\epsilon \to 0^+} (\begin{pmatrix} L_{11}^T Q_{11}^{-1} L_{11}/\epsilon + L_{11}^T Q_{11}^{-1} Q_{12} \tilde{R}^{-1} Q_{21} Q_{11}^{-1} L_{11} & -L_{11}^T Q_{11}^{-1} Q_{12} \tilde{R}^{-1} L_{22} \\ -L_{22}^T \tilde{R}^{-1} Q_{21} Q_{11}^{-1} L_{11} & L_{22}^T \tilde{R}^{-1} L_{22} \end{pmatrix} + o(\epsilon))^{-1} \\
&= \lim_{\epsilon \to 0^+} \begin{pmatrix} L_{11}^T Q_{11}^{-1} L_{11}/\epsilon + L_{11}^T Q_{11}^{-1} Q_{12} \tilde{R}^{-1} Q_{21} Q_{11}^{-1} L_{11} & -L_{11}^T Q_{11}^{-1} Q_{12} \tilde{R}^{-1} L_{22} \\ -L_{22}^T \tilde{R}^{-1} Q_{21} Q_{11}^{-1} L_{11} & L_{22}^T \tilde{R}^{-1} L_{22} \end{pmatrix}^{-1} \\
&\stackrel{\text{def}}{=} \lim_{\epsilon \to 0^+} (\begin{pmatrix} X/\epsilon + V & Y \\ Y^T & Z \end{pmatrix}^{-1} \qquad \text{(A.9)}
\end{aligned}
$$

where X and Z are both positive-definite since L_{11} and L_{22} have full rank. Applying the identity (A.6) to (A.9) we get

$$
\begin{aligned}
P_{ML} &= \lim_{\epsilon \to 0^+} \begin{pmatrix} \epsilon X^{-1} & -\epsilon X^{-1} Y (Z - \epsilon Y^T X^{-1} Y)^{-1} \\ -\epsilon (Z - \epsilon Y^T X^{-1} Y)^{-1} Y^T X^{-1} & (Z - \epsilon Y^T X^{-1} Y)^{-1} \end{pmatrix} + o(\epsilon^2) \\
&= \begin{pmatrix} 0 & 0 \\ 0 & Z^{-1} \end{pmatrix} . \qquad \text{(A.10)}
\end{aligned}
$$

Thus,

$$
P_{ML} = \begin{pmatrix} 0 & 0 \\ 0 & (L_{22}^T \tilde{R}^{-1} L_{22})^{-1} \end{pmatrix} . \qquad \text{(A.11)}
$$

By a similar argument, expression (2.29) yields

$$\hat{x}_{ML} = \begin{pmatrix} (L_{11}^T Q_{11}^{-1} L_{11})^{-1} L_{11}^T Q_{11}^{-1} & 0 \\ 0 & (L_{22}^T \tilde{R}^{-1} L_{22})^{-1} L_{22}^T \tilde{R}^{-1} \end{pmatrix} z. \tag{A.12}$$

By noting that $(L_{11}^T Q_{11}^{-1} L_{11})^{-1} L_{11}^T Q_{11}^{-1}$ is a left inverse of L_{11}, we can see that (A.12) is consistent with (2.25) and thus the lemma is proved. □

Note that the non-unicity in the expression (A.3) which is due to the fact that Q_{11} can be any positive-definite matrix, is related to the non-unicity of the left inverse of L_{11} when L_{11} is not square, i.e., when redundant, perfect observations are available.

B Proof of Theorem 3.1

Let

$$\chi(k)^T = \begin{pmatrix} x(0)^T & x(1)^T & \dots & x(k)^T \end{pmatrix}^T \tag{B.1}$$

$$\eta(k)^T = \begin{pmatrix} \bar{x}_0^T & y(1)^T & \dots & y(k)^T \end{pmatrix}^T \tag{B.2}$$

$$\mu(k)^T = \begin{pmatrix} u(0)^T & u(1)^T & \dots & u(k-1)^T \end{pmatrix}^T \tag{B.3}$$

$$\rho(k)^T = \begin{pmatrix} \nu^T & r(0)^T & \dots & r(k-1)^T \end{pmatrix}^T, \tag{B.4}$$

and consider the problem of estimating $\chi(k)$ based on the following observations

$$\mathcal{S}_k \chi(k) = \mathcal{B}_k \mu(k) \tag{B.5}$$

$$\eta(k) = \mathcal{C}_k \chi(k) + \rho(k) \tag{B.6}$$

where

$$\Sigma_k = \begin{pmatrix} -A_0 & E_1 & & & & \\ & -A_1 & E_2 & & & \\ & & .. & .. & & \\ & & & .. & .. & \\ & & & & -A_{k-1} & E_k \end{pmatrix} \tag{B.7}$$

$$\mathcal{C}_k = \begin{pmatrix} I & & & & \\ & C_1 & & & \\ & & \dots & & \\ & & & \dots & \\ & & & & C_k \end{pmatrix}. \tag{B.8}$$

This estimation problem is well-posed because $\begin{pmatrix} \mathcal{S}_k \\ \mathcal{C}_k \end{pmatrix}$ has full column rank (that thanks to the assumption that $\begin{pmatrix} E_k \\ C_k \end{pmatrix}$ has full column rank).

It is easy to see that

$$\hat{x}(k) = \begin{pmatrix} 0 & .. & 0 & I \end{pmatrix} \hat{\chi}(k) \tag{B.9}$$

$$P_k = \begin{pmatrix} 0 & .. & 0 & I \end{pmatrix} \mathcal{P}_k \begin{pmatrix} 0 \\ \vdots \\ 0 \\ I \end{pmatrix} \tag{B.10}$$

where $\hat{\chi}(k)$ denote the ML estimate of $\chi(k)$ based on observations (B.5) and (B.6) and $\mathcal{P}_k$ the corresponding estimation error variance.

From the results of the previous section, we know that

$$\hat{\chi}(k) = \lim_{\epsilon \to 0+} Z_k(\epsilon) \begin{pmatrix} \Sigma_k^T & \mathcal{C}_k^T \end{pmatrix} \left(\begin{pmatrix} \mathcal{Q}_k & \mathcal{S}_k \\ \mathcal{S}_k^T & \mathcal{R}_k \end{pmatrix} + \epsilon I \right)^{-1} \begin{pmatrix} 0 \\ I \end{pmatrix} \eta(k) \tag{B.11}$$

where

$$Z_k(\epsilon) = \left[\begin{pmatrix} \Sigma_k^T & \mathcal{C}_k^T \end{pmatrix} \left(\begin{pmatrix} \mathcal{Q}_k & \mathcal{S}_k \\ \mathcal{S}_k^T & \mathcal{R}_k \end{pmatrix} + \epsilon I \right)^{-1} \begin{pmatrix} \Sigma_k \\ \mathcal{C}_k \end{pmatrix} \right]^{-1} \tag{B.12}$$

and where

$$\mathcal{R}_k = \begin{pmatrix} P_0 & & & & \\ & R_0 & & & \\ & & R_2 & & \\ & & & \cdots & \\ & & & & R_{k-1} \end{pmatrix} \tag{B.13}$$

$$\mathcal{Q}_k = \begin{pmatrix} Q_0 & & & \\ & Q_1 & & \\ & & \cdots & \\ & & & Q_{k-1} \end{pmatrix} \tag{B.14}$$

$$\mathcal{S}_k = \begin{pmatrix} 0 & S_0 & & & \\ 0 & 0 & S_1 & & \\ & & & \cdots & \\ & & & & S_{k-1} \end{pmatrix} \tag{B.15}$$

Note that

$$\mathcal{S}_{j+1} = \begin{pmatrix} \mathcal{S}_j & 0 \\ \begin{pmatrix} 0 & .. & 0 & -A_j \end{pmatrix} & E_{j+1} \end{pmatrix} \tag{B.16}$$

$$\mathcal{C}_{j+1} = \begin{pmatrix} \mathcal{C}_j & 0 \\ 0 & C_{j+1} \end{pmatrix} \tag{B.17}$$

$$\mathcal{Q}_{j+1} = \begin{pmatrix} \mathcal{Q}_j & 0 \\ 0 & Q_j \end{pmatrix} \tag{B.18}$$

$$\mathcal{R}_{j+1} = \begin{pmatrix} \mathcal{R}_j & 0 \\ 0 & R_{j+1} \end{pmatrix}. \tag{B.19}$$

And thus

$$Z_{j+1}(\epsilon) = \left[\begin{pmatrix} \Sigma_{j+1}^T & \mathcal{C}_{j+1}^T \end{pmatrix} \left(\begin{pmatrix} \mathcal{Q}_{j+1} & \mathcal{S}_{j+1} \\ \mathcal{S}_{j+1}^T & \mathcal{R}_{j+1} \end{pmatrix} + \epsilon I \right)^{-1} \begin{pmatrix} \Sigma_{j+1} \\ \mathcal{C}_{j+1} \end{pmatrix} \right]^{-1} =$$

$$\begin{pmatrix} Z_j(\epsilon)^{-1} + \begin{pmatrix} 0 & .. & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & .. & 0 & \Omega_1(\epsilon) \end{pmatrix} & \begin{pmatrix} 0 \\ \vdots \\ \Omega_2(\epsilon) \end{pmatrix} \\ \begin{pmatrix} 0 & \dots & 0 & \Omega_2(\epsilon)^T \end{pmatrix} & \Omega_3(\epsilon) \end{pmatrix}^{-1} \tag{B.20}$$

where

$$\Omega_1(\epsilon) = \begin{pmatrix} A_j^T & 0 \end{pmatrix} \left(\begin{pmatrix} Q_j & S_j \\ S_j^T & R_j \end{pmatrix} + \epsilon I \right)^{-1} \begin{pmatrix} A \\ 0 \end{pmatrix} \tag{B.21}$$

$$\Omega_2(\epsilon) = \begin{pmatrix} A_j^T & 0 \end{pmatrix} \left(\begin{pmatrix} Q_j & S_j \\ S_j^T & R_j \end{pmatrix} + \epsilon I \right)^{-1} \begin{pmatrix} E_{j+1} \\ C_{j+1} \end{pmatrix} \tag{B.22}$$

$$\Omega_3(\epsilon) = \begin{pmatrix} E_{j+1}^T & C_{j+1}^T \end{pmatrix} \left(\begin{pmatrix} Q_j & S_j \\ S_j^T & R_j \end{pmatrix} + \epsilon I \right)^{-1} \begin{pmatrix} E_{j+1} \\ C_{j+1} \end{pmatrix}. \tag{B.23}$$

If we denote the (j,j)-block entry of $Z_j(\epsilon)$ by $P_j(\epsilon)$ and use the matrix identity (A.6), and the matrix identity

$$(A + BCD)^{-1} = A^{-1} - A^{-1}B(C^{-1} + DA^{-1}B)^{-1}DA^{-1} \tag{B.24}$$

after some algebra, (B.20) yields

$$Z_{j+1}(\epsilon) =$$

$$\begin{pmatrix} * & * \\ \begin{pmatrix} 0 & .. & 0 & T_{j+1}^{-1}(\epsilon) \begin{pmatrix} E_{j+1}^T & C_{j+1}^T \end{pmatrix} \left(\begin{pmatrix} A_j P_j(\epsilon) A_j + Q_j & S_j \\ S_j^T & R_j \end{pmatrix} + \epsilon I \right)^{-1} \begin{pmatrix} A_j \\ 0 \end{pmatrix} \end{pmatrix} Z_j(\epsilon) & T_{j+1}^{-1}(\epsilon) \end{pmatrix} \tag{B.25}$$

where $*$'s denote "don't care entries" and

$$T_{j+1}(\epsilon) = \begin{pmatrix} E_{j+1}^T & C_{j+1}^T \end{pmatrix} \left(\begin{pmatrix} A_j P_j(\epsilon) A_j + Q_j & S_j \\ S_j^T & R_j \end{pmatrix} + \epsilon I \right)^{-1} \begin{pmatrix} E_{j+1} \\ C_{j+1} \end{pmatrix}. \tag{B.26}$$

It is not difficult to see that $Z_j(\epsilon)$ and thus $P_j(\epsilon)$ are increasing in $\epsilon > 0$,[6] and since $P_j(\epsilon)$ goes to P_j as ϵ goes to zero, we have

$$P_j(\epsilon) = P_j + \epsilon\Lambda + o(\epsilon^2) \tag{B.27}$$

where $\Lambda \geq 0$.

Finally, noting that

$$\hat{\chi}(j+1) = \lim_{\epsilon \to 0^+} Z_{j+1}(\epsilon) \begin{pmatrix} \Sigma_{j+1}^T & C_{j+1}^T \end{pmatrix} \left(\begin{pmatrix} \mathcal{Q}_{j+1} & \mathcal{S}_{j+1} \\ \mathcal{S}_{j+1}^T & \mathcal{R}_{j+1} \end{pmatrix} + \epsilon I \right)^{-1} \begin{pmatrix} 0 \\ I \end{pmatrix} \eta(j+1) \tag{B.28}$$

we get that

$$\hat{x}(j+1) = \lim_{\epsilon \to 0^+} \left[\begin{pmatrix} E_{j+1}^T & C_{j+1}^T \end{pmatrix} \left(\begin{pmatrix} Q_{j+1} & S_{j+1} \\ S_{j+1}^T & R_{j+1} \end{pmatrix} + \epsilon \begin{pmatrix} I + A_j \Lambda A_j^T & \\ 0 & I \end{pmatrix} \right)^{-1} \begin{pmatrix} \Sigma_{j+1} \\ C_{j+1} \end{pmatrix} \right]^{-1}$$

[6] The derivative of $Z_j(\epsilon)$, $\dot{Z}_j(\epsilon) = \epsilon Z_j(\epsilon) \left[\begin{pmatrix} \Sigma_j^T & C_j^T \end{pmatrix} \left(\begin{pmatrix} \mathcal{Q}_j & \mathcal{S}_j \\ \mathcal{S}_j^T & \mathcal{R}_j \end{pmatrix} + \epsilon I \right)^{-2} \begin{pmatrix} \Sigma_j \\ C_j \end{pmatrix} \right] Z_j(\epsilon) \geq 0$, when $\epsilon \geq 0$.

$$\begin{pmatrix} \Sigma_{j+1}^T & C_{j+1}^T \end{pmatrix} \left(\begin{pmatrix} Q_{j+1} & S_{j+1} \\ S_{j+1}^T & R_{j+1} \end{pmatrix} + \epsilon \begin{pmatrix} I + A_j \Lambda A_j^T & \\ 0 & I \end{pmatrix} \right)^{-1} \begin{pmatrix} A_j \hat{x}_j \\ y(j+1) \end{pmatrix} \tag{B.29}$$

$$P_{j+1} = \lim_{\epsilon \to 0+} \left[\begin{pmatrix} \Sigma_{j+1}^T & C_{j+1}^T \end{pmatrix} \left(\begin{pmatrix} Q_{j+1} & S_{j+1} \\ S_{j+1}^T & R_{j+1} \end{pmatrix} + \epsilon \begin{pmatrix} I + A_j \Lambda A_j^T & \\ 0 & I \end{pmatrix} \right)^{-1} \begin{pmatrix} \Sigma_{j+1} \\ C_{j+1} \end{pmatrix} \right]^{-1} \tag{B.30}$$

The Theorem is now proved because (B.29) and (B.30) are consistent with (2.29) and (2.28).

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